Chapter 5

Distribution Theory

5.1 Test functions on $\mathbb{R}^n$

We will consider three spaces of test functions: compactly supported smooth, Schwartz, and smooth functions, respectively:

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n). \tag{5.1.1}$$

We will often omit writing $\mathbb{R}^n$. Now let us define the notion of convergence in each of these spaces:

Definition 5.1. *(i) Let $\phi_n, \phi \in C_c^\infty$. We say that $\phi_n \to \phi$* ***in*** *C_c^∞ iff there exists a compact set K containing the supports of all ϕ_n and for any multi-index α:*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^n} |\partial^\alpha(\phi_n - \phi)(x)| = 0.$$

(ii) Let $\phi_n, \phi \in \mathcal{S}$. We say that $\phi_n \to \phi$ ***in*** *$\mathcal{S}$ iff for all multi-indices α, β:*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta(\phi_n - \phi)(x)| = 0.$$

(iii) Let $\phi_n, \phi \in C^\infty$. We say that $\phi_n \to \phi$ ***in*** *C^∞ iff for all $N \in \mathbb{N}$ and any multi-index α:*

$$\lim_{n\to\infty} \sup_{|x|\le N} |\partial^\alpha(\phi_n - \phi)(x)| = 0.$$

Remarks 5.2. (a) Convergence in C_c^∞ implies convergence in $\mathcal{S}$, which implies convergence in C^∞. For example, let $\phi \in C_c^\infty$. Then sequence $\phi_n(x) := \phi(x-n)/n$ converges to 0 in C^∞, but it has no limit in $\mathcal{S}$ or in C_0^∞.

(b) As we discussed earlier in the course, $\mathcal{S}$ is a Fréchet (i.e., complete) space with semi-norms given by (4.2.1). It can be shown that C^∞ is also a Fréchet space with the semi-norms given by $||\phi||_{\alpha,N} := \sup_{|x|\le N} |(\partial^\alpha\phi)(x)|$. However this family of semi-norms would not make C_c^∞ into a Fréchet space. That is why it is more natural to induce the topology from the subspaces $C^\infty(K_n)$ for a sequence of compacts K_n that exhaust $\mathbb{R}^n$ (each of which is a Fréchet space). This explains the requirement of existence of a compact K, and this construction makes $C_c(\mathbb{R}^n)$ into a complete space.

5.2 Distributions, tempered distributions, and distributions with compact support: definitions

Definition 5.3. *(i) Define the* ***space of distributions*** $\mathcal{D}'(\mathbb{R}^n)$ *to be* $(C_c^\infty(\mathbb{R}^n))'$*, the dual of* $C_c^\infty(\mathbb{R}^n)$ *(that is, the space of continuous linear functionals* $C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$*).*

(ii) Define the ***space of tempered distributions*** $\mathcal{S}'(\mathbb{R}^n)$ *to be* $(\mathcal{S}(\mathbb{R}^n))'$*, the dual of* $\mathcal{S}(\mathbb{R}^n)$ *(that is, the space of continuous linear functionals* $\mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$*).*

(iii) Define the ***space of distributions with compact support*** $\mathcal{E}'(\mathbb{R}^n)$ *to be* $(C^\infty(\mathbb{R}^n))'$*, the dual of* $C^\infty(\mathbb{R}^n)$ *(that is, the space of continuous linear functionals* $C^\infty(\mathbb{R}^n) \to \mathbb{C}$*).*

The topology on each of these spaces are defined to be pointwise (i.e., weak-∗) topology: $T_k \to T$ *in* $\mathcal{D}'$ *iff* $T_k(\phi) \to T(\phi)$ *for every* $\phi \in C_c^\infty$*, and similarly for* $\mathcal{S}'$ *and* $\mathcal{E}'$*.*

Remarks 5.4. (a) Due to (5.1.1), we have the inclusions

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n). \tag{5.2.1}$$

(b) Let T be a distribution and ϕ a test function. Instead of writing $T(\phi)$ we will instead usually write $\langle T, \phi\rangle$.

If $f \in \mathcal{D}'$ and ϕ is a test function, instead of using the notation $f(\phi)$, we will start writing $\langle f, \phi\rangle$.

Proposition 5.5. *(i) A linear functional* f *on* $C_c^\infty(\mathbb{R}^n)$ *is continuous (i.e., is a distribution* $f \in \mathcal{D}'$*) iff for any compact* $K \subset \mathbb{R}^n$ *there is* $C > 0$ *and* $m \in \mathbb{Z}$ *such that*

$$|\langle f, \phi\rangle| \le C \sum_{|\alpha|\le m} ||\partial^\alpha \phi||_\infty \qquad \text{for all } \phi \in C_c^\infty \text{ with support in } K.$$

(ii) A linear functional f *on* $\mathcal{S}(\mathbb{R}^n)$ *is continuous (i.e., is a tempered distribution* $f \in \mathcal{S}'$*) iff there is* $C > 0$ *and* $k, m \in \mathbb{Z}$ *such that*

$$|\langle f, \phi\rangle| \le C \sum_{\substack{|\alpha|\le m \\ |\beta|\le k}} ||x^\alpha \partial^\beta \phi||_\infty \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n). \tag{5.2.2}$$

(iii) A linear functional f *on* $C^\infty(\mathbb{R}^n)$ *is continuous (i.e., is a distribution with compact support* $f \in \mathcal{E}'$*) iff there is* $C > 0$ *and* $N, m \in \mathbb{Z}$ *such that*

$$|\langle f, \phi\rangle| \le C \sum_{|\alpha|\le m} \sup_{|x|\le N} |(\partial^\alpha \phi)(x)| \qquad \text{for all } \phi \in C^\infty.$$

Proof. Let us prove (ii). That (5.2.2) implies continuity is clear: $\phi_n \to f$ in each $||\cdot||_{\alpha,\beta}$ semi-norm implies that $\langle f, \phi_n - \phi\rangle \to 0$ by (5.2.2). Let us now prove the converse. Suppose f is continuous. Recall that semi-norms $||\phi||_{\alpha,\beta} := ||x^\alpha \partial^\beta \phi||_\infty$ generate the topology on $\mathcal{S}$, which means that the family of sets $\{f \in \mathcal{S} : ||f||_{\alpha,\beta} < \delta\}$ for various α, β, and $\delta > 0$, forms a subbasis for the topology of $\mathcal{S}$. Thus if f is continuous, then $f^{-1}(\{z \in \mathbb{C} : |z| < 1\})$ is open so contains an element of the base, i.e., a finite intersection of $\{\phi \in \mathcal{S} : ||\phi||_{\alpha_j,\beta_j} < \delta_j\}$. Take $\delta := \min \delta_j$, $m = \max|\alpha_j|$ and $k = \max|\beta_j|$. Then if ϕ has $||\phi||_{\alpha,\beta} < \delta$ for all $|\alpha| \le m$ and $|\beta| \le k$, we have $|\langle f, \phi\rangle| < 1$. Finally, use linearity: for arbitrary $\psi \in \mathcal{S}$, define $\phi := \delta\psi / \sum_{|\alpha|\le m, |\beta|\le k} ||\psi||_{\alpha,\beta}$. Then from what we just proved, $|\langle f, \psi\rangle| < 1$. This implies (5.2.2) with $C = 1/\delta$. ☐

5.3 Examples of distributions

Example 5.6. Any $f \in L^p$, $1 \le p \le \infty$ is a distribution, $f \in \mathcal{D}'$, where we define its action on a test function $\phi \in C_c^\infty$ via

$$\langle f, \phi \rangle := \int_{\mathbb{R}^n} g(x)\phi(x)\,dx \tag{5.3.1}$$

Note that $\phi_n \to \phi$ in C_c^∞ implies $\langle f, \phi_n \rangle \to \langle f, \phi \rangle$. We will denote such a distribution by the same symbol f. This is where notation $\langle f, \phi \rangle$ pays off compared to the usual $f(\phi)$ which would be confusing together with $f(x)$.

Any $f \in L^p$, $1 \le p \le \infty$, is, in fact, also a tempered distribution, $f \in \mathcal{S}'$, by the same logic.

However, $f \in L^p$, $1 \le p \le \infty$, is not in general in $\mathcal{E}'$ unless f has a compact support.

Example 5.7. More generally, any locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ (meaning $\int_K |f|\,dx < \infty$ for any compact K; note that $L^1_{loc}(\mathbb{R}^n)$ includes any $L^p(\mathbb{R}^n)$) also defines a distribution, $f \in \mathcal{D}'$ (but not necessarily in $\mathcal{S}'$ or $\mathcal{E}'$).

For example, the function $e^{|x|^2}$ is in $\mathcal{D}'$ but not in $\mathcal{S}'$ or $\mathcal{E}'$. The function 1 is in $\mathcal{D}'$ and $\mathcal{S}'$ but not in $\mathcal{E}'$.

Example 5.8. Any finite Borel measure μ is in $\mathcal{D}'$ and in $\mathcal{S}'$ if we define for a test function f:

$$\langle \mu, \phi \rangle := \int_{\mathbb{R}^n} \phi(x)\,d\mu(x).$$

Indeed, if $\phi_n \to \phi$ in $\mathcal{S}$ or in C_c^∞, then $\langle \mu, \phi_n \rangle \to \langle \mu, \phi \rangle$.

In general finite Borel measure do not have to be in $\mathcal{E}'$.

Example 5.9. Some non-finite Borel measures may also be in $\mathcal{S}'$, for example the Lebesgue measure is a tempered distribution.

Example 5.10. The Dirac point measure at the origin δ_0 is an example of a finite Borel measure. As such, it is a tempered distribution with its action on $\phi \in \mathcal{S}$ defined via $\langle \delta_0, \phi \rangle := \int \phi\,d\delta_0 = \phi(0)$. Moreover, it is in fact of compact support, $\delta_0 \in \mathcal{E}'$ since $\langle \delta_0, \phi_n \rangle \to \langle \delta_0, \phi \rangle$ whenever $\phi_n \to \phi$ in C^∞.

Example 5.11. Any function f that satisfies $|f(x)| \le C(1+|x|)^k$ for some real k is a tempered distribution as it can be integrated against any Schwartz function.

$\log|x|$ is also a tempered distribution: even though it is not well defined at $x = 0$, it can be integrated against any Schwartz function.

Example 5.12. An example of a distribution $f \in \mathcal{D}'$ that isn't a function nor a measure is the map $\phi \mapsto \partial^\alpha f(x_0)$ (this also belongs to $\mathcal{S}'$ and $\mathcal{E}'$, of course).

5.4 Operations on distributions: distributional derivative

Intuition. Let us discuss operations on distributions. Addition and scalar multiplication are of course immediately clear from the vector space structure. Warning: product of two distributions is not defined in general!

One of the strengths of the distributions is that we can naturally define their derivatives. This notion generalizes the usual notion of derivative for those distributions that come from the smooth functions.

The idea behind the definition below comes from the integration by parts: suppose a distribution f is given by a smooth function, and let ϕ be a test function, then

$$\langle f', \phi \rangle = \int f'(x)\phi(x)\,dx = -\int f(x)\phi'(x)\,dx \tag{5.4.1}$$

by the integration by parts (the boundary terms vanish).

Definition 5.13. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$. For any multi-index α, define the* ***(distributional) derivative*** $\partial^\alpha f \in \mathcal{D}'(\mathbb{R}^n)$ *of f via*

$$\langle \partial^\alpha f, \phi \rangle := (-1)^{|\alpha|} \langle f, \partial^\alpha \phi \rangle \qquad \text{for any } \phi \in C_c^\infty.$$

Remarks 5.14. (a) If $f \in \mathcal{S}'$, one defines $\partial^\alpha f \in \mathcal{S}'$ in the exact same way but using test functions in $\mathcal{S}$ instead of C_c^∞. Similarly for $f \in \mathcal{E}'$.

(b) In particular, any locally integrable function has a distributional derivative (which need not have to be a locally integrable function however).

Examples 5.15. (a) Because of the integration by parts (5.4.1), if f is a smooth function, then its usual derivative and distributional derivative agree.

(b) If $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 0$ (so that $H \in \mathcal{S}'$), then

$$\langle H'(x), \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)\, dx = \phi(x)|_0^\infty = \phi(0) = \langle \delta_0, \phi \rangle,$$

so that H' is the Dirac mass point at 0.

(c) It is clear that the derivative of δ_0 is the distribution given by $\langle \delta_0', \phi \rangle = -\langle \delta_0, \phi' \rangle = -\phi'(0)$.

5.5 Operations on distributions: Fourier and inverse Fourier transforms

Intuition. Recall that for $f, \phi \in \mathcal{S}(\mathbb{R}^n)$ we had the property $\int \widehat{f}\phi\, dx = \int f\widehat{\phi}\, dx$ (in fact this holds for $f, \phi \in L^1(\mathbb{R}^n)$. It is therefore natural to use this as a motivation for the definition below.

Definition 5.16. *Let f be a tempered distribution $f \in \mathcal{S}'$. Then we define the* ***Fourier transform of*** *f to be the tempered distribution $\widehat{f} \in \mathcal{S}'$ given by*

$$\left\langle \widehat{f}, \phi \right\rangle := \left\langle f, \widehat{\phi} \right\rangle, \qquad \text{for any } \phi \in \mathcal{S}. \tag{5.5.1}$$

The ***inverse Fourier transform of*** *$f \in \mathcal{S}'$ is defined to be $f^\vee \in \mathcal{S}'$ given by*

$$\langle f^\vee, \phi \rangle := \langle f, \phi^\vee \rangle, \qquad \text{for any } \phi \in \mathcal{S}.$$

Remarks 5.17. (a) If $f \in \mathcal{E}'$, we can define its Fourier transform by viewing $f \in \mathcal{E}' \subset \mathcal{S}'$: i.e., we are still using (5.5.1) with test functions $f \in \mathcal{S}$. In particular $\widehat{f} \in \mathcal{S}'$ and not necessarily $\widehat{f} \in \mathcal{E}'$. In fact, one can show that $\widehat{f}$ is can be represented by a slowly increasing C^∞ function, see [F, 9.11].

(b) If $f \in \mathcal{D}' \setminus \mathcal{S}'$, then we would need (5.5.1) for $\phi \in C_c^\infty$, but $\widehat{\phi}$ is then not in C_c^∞, so this breaks down.

Examples 5.18. (a) Notice that $\widehat{\delta_0} = 1$. More generally, $\widehat{\delta_{x_0}} = e^{-2\pi i x \cdot x_0}$ for any $x_0 \in \mathbb{R}^n$. Indeed,

$$\left\langle \widehat{\delta_{x_0}}, \phi \right\rangle = \left\langle \delta_{x_0}, \widehat{\phi} \right\rangle = \widehat{\phi}(x_0) = \int_{\mathbb{R}^n} \phi(x) e^{-2\pi i x \cdot x_0}\, dx.$$

(b) More generally, $(\partial^\alpha \delta_{x_0})^\wedge = (2\pi i x)^\alpha e^{-2\pi i x \cdot x_0}$.

(c) Even more generally, Fourier transforms of the linear combinations of delta functions and their derivatives are precisely the trigonometric polynomials.

(d) Conversely, applying the inverse Fourier transform, we get $(x^\alpha)^\wedge = (-2\pi i)^{-|\alpha|} \partial^\alpha \delta_0$.

5.6 Operations on distributions: translations and reflections

Recall the notation $\widetilde{\phi}(x) := \phi(-x)$ and $\tau_y(\phi)(x) := \phi(x-y)$ for functions. Let us also define the dilation $\delta^a(\phi) := \phi(ax)$ for $a > 0$. We can extend these to the distributions:

Definition 5.19. *Let $f \in \mathcal{D}'$.*

(i) For $t \in \mathbb{R}^n$, define the ***translation of*** *f* ***by*** *t to be the distribution $\tau_t(f) \in \mathcal{D}'$ given by*

$$\langle \tau_t(f), \phi \rangle = \langle f, \tau_{-t}(\phi) \rangle .$$

(ii) The ***reflection of*** *f is the distribution $\widetilde{f} \in \mathcal{D}'$ given by*

$$\left\langle \widetilde{f}, \phi \right\rangle = \left\langle f, \widetilde{\phi} \right\rangle .$$

(iii) For $a > 0$, the ***dilation of*** *f is the distribution $\delta^a(f) \in \mathcal{D}'$ given by*

$$\langle \delta^a(f), \phi \rangle = \left\langle f, a^{-n}\delta^{1/a}(\phi) \right\rangle .$$

Remark 5.20. (a) These formulas are of course motivated by a simple change of variable.

(b) It is easy to see that these operations preserve spaces $\mathcal{S}'$ and $\mathcal{E}'$.

(c) For example, $\tau_t(\delta_0) = \delta_t$; $\widetilde{\delta_x} = \delta_{-x}$ and $\delta^a(\delta_0) = a^{-n}\delta_0$.

5.7 Operations on distributions: convolutions

Intuition. Note that if $f, g, h \in \mathcal{S}$, then by Fubini's theorem

$$\int_{\mathbb{R}^n} (h * g)(x) f(x)\, dx = \int\int h(x-y)g(y)\, dy f(x)\, dx = \int_{\mathbb{R}^n} g(x)(\widetilde{h} * f)(x) dx.$$

Definition 5.21. *Suppose $f \in \mathcal{S}'$ and $h \in \mathcal{S}$. Define the* ***convolution of*** *f* ***and*** *h as a tempered distribution $f * h \in \mathcal{S}'$ given by*

$$\langle f * h, \phi \rangle := \left\langle f, \widetilde{h} * \phi \right\rangle , \qquad \phi \in \mathcal{S}.$$

Remarks 5.22. (a) In this case ($f \in \mathcal{S}'$ and $h \in \mathcal{S}$) one can show that $f * h$ is a C^∞ function.

(b) In general convolution of two tempered distribution does not have to exist.

(c) For $f \in \mathcal{E}'$, one can define the convolution of f and $h \in C^\infty$. Similarly can be defined the convolution of $f \in \mathcal{D}'$ and $h \in C_c^\infty$. It can be shown that $f * h$ is a C^∞ function in each of these cases.

(d) Moreover, one can in fact define the convolution of $f \in \mathcal{D}'$ and $h \in \mathcal{E}'$. Then $f * h = h * f \in \mathcal{D}'$. We skip the proofs.

5.8 Operations on distributions: products

Definition 5.23. *Let $f \in \mathcal{S}'$ and $h \in C^\infty$ such that $|(\partial^\alpha h)(x)| \le C(1+|x|)^{k_\alpha}$ for all α and some $k_\alpha > 0$. Then we can define the* ***product of*** *f* ***and*** *h to be the tempered distribution $fh \in \mathcal{S}'$ given by*

$$\langle fh, \phi \rangle = \langle f, h\phi \rangle , \qquad \phi \in \mathcal{S}.$$

Remark 5.24. In other words, product of two distributions can be defined only in special situations.

5.9 Properties of the distributional Fourier transform

Proposition 5.25. *Let $f, g \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then for all $y \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$:*

(i) $\widehat{f+g} = \widehat{f} + \widehat{g}$;

(ii) $\widehat{\lambda f} = \lambda \widehat{f}$;

(iii) $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$;

(iv) $\widehat{\tau_y(f)} = \widehat{f} e^{-2\pi i \xi \cdot y}$;

(v) $\left(e^{2\pi i y \cdot x} f\right)\widehat{} = \tau_y(\widehat{f})$;

(vi) $\widehat{f * \psi} = \widehat{f}\widehat{\psi}$;

(vii) $\widehat{f\psi} = \widehat{f} * \widehat{\psi}$;

(viii) $\widehat{\partial^\alpha f} = (2\pi i \xi)^\alpha \widehat{f}$;

(ix) $\partial^\alpha(\widehat{f}) = ((-2\pi i x)^\alpha f)\widehat{}$;

(x) $(\widehat{f})^\vee = f$.

5.10 Support of a distribution

Definition 5.26. *For $f \in \mathcal{D}'(\mathbb{R}^n)$ we define the **support of** f, $\operatorname{supp} f$, to be the intersection of all closed sets K with the property*

$$\phi \in C^\infty(\mathbb{R}^n), \operatorname{supp}\phi \subseteq K^c \Rightarrow \langle f, \phi \rangle = 0.$$

Proposition 5.27. *$f \in \mathcal{D}'$ has compact support iff $f \in \mathcal{E}'$.*

Proof. Suppose $f \in \mathcal{E}'$. Then the inequality in Proposition 5.5(iii) holds for some C, N, m. This means that if $\operatorname{supp}\phi \subset \{|x| > N\}$ then $\langle f, \phi \rangle = 0$. So $\operatorname{supp} f$ is bounded and therefore compact as it is closed by definition.

Conversely, suppose $f \in \mathcal{D}'$ and $\operatorname{supp} f$ is compact. Thus $\operatorname{supp} f \subseteq \{x : |x| \leq N\}$ for some N. Choose $\psi \in C_c^\infty$ such that $\psi(x) = 1$ on $\{x : |x| \leq N\}$ and $\psi(x) = 0$ on $\{x : |x| \geq N+1\}$. Then for any $\phi \in C^\infty$, we have that $\phi - \phi\psi$ has no support in $|x| \leq N$, so $\langle f, \phi \rangle = \langle f, \phi\psi \rangle$. Now by continuity of f, we can take $K = \{x : |x| \leq N+1\}$ in Proposition 5.5(i) to get

$$\langle f, \phi \rangle = \langle f, \phi\psi \rangle \leq C \sum_{|\alpha| \leq m} ||\partial^\alpha(\phi\psi)||_\infty$$

It is easy to see that the right-hand side of this expression can be bounded by $\tilde{C} \sum_{|\alpha| \leq m} \sup_{|x| \leq N+1} |(\partial^\alpha \phi)(x)|$. This proves Proposition 5.5(iii), i.e., $f \in \mathcal{E}'$. $\square$

Example 5.28. It is easy to see that δ_{x_0} has $\{x_0\}$ as its support. Moreover, any of the derivatives of δ_{x_0} has $\{x_0\}$ as the support. What about the converse?

Proposition 5.29. *Suppose $f \in \mathcal{S}'$ has $\operatorname{supp} f = \{x_0\}$. Then there exists an integer m and complex numbers c_α such that*

$$f = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta_{x_0}. \tag{5.10.1}$$

Proof. Without loss of generality, assume $x_0 = 0$.

Let $f \in \mathcal{S}'$ has $\operatorname{supp} f = \{0\}$. By Proposition 5.5(ii), there are some C, m, k such that

$$|\langle f, \phi \rangle| \leq C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq k}} ||x^\alpha \partial^\beta \phi||_\infty \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n). \tag{5.10.2}$$

Let $\phi \in \mathcal{S}$. We are heading towards proving that $\langle f, \phi \rangle$ is equal to $\sum_{|\alpha| \leq k} c_\alpha \partial^\alpha \phi(0)$ for some constants c_α's. Let us apply Taylor's theorem to write $\phi(x) = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} (\partial^\alpha \phi)(0) x^\alpha + h(x)$, where $h \in O(x^{k+1})$ as $x \to 0$. Note that $h \notin \mathcal{S}$, so we cannot apply f term-by-term. To fix this, let us introduce $\eta \in C_c^\infty$ that is 1 in a neighbourhood of 0. Then

$$\phi(x) = \eta(x) \left(\sum_{|\alpha| \leq k} \tfrac{1}{\alpha!} (\partial^\alpha \phi)(0) x^\alpha \right) + \eta(x) h(x) + (1 - \eta(x)) \phi(x). \tag{5.10.3}$$

Now note that $(1 - \eta)\phi$ has support in $\mathbb{R} \setminus \{x_0\}$, so $\langle f, (1-\eta)\phi \rangle = 0$ by the definition of $\operatorname{supp} f = \{0\}$. We also claim that $\langle f, \eta h \rangle = 0$: rewrite $\eta(x)h(x) = \eta(x)h(x)\eta(x/\varepsilon) + \eta(x)h(x)(1 - \eta(x/\varepsilon))$; then $\langle f, \eta(x)h(x)(1 - \eta(x/\varepsilon)) \rangle = 0$ by the same support argument as before, and the $||x^\alpha \partial^\beta (\cdot)||_\infty$ semi-norms of $\eta(x)h(x)\eta(x/\varepsilon)$ go to 0 for all $|\alpha| \leq m$ and $|\beta| \leq k$ because $\partial^\beta h(0) = 0$ for all $|\beta| \leq k$. So by (5.10.2), $\langle f, \eta h \rangle = 0$.

So we conclude that

$$\langle f, \phi \rangle = \sum_{|\alpha| \leq k} \tfrac{1}{\alpha!} (\partial^\alpha \phi)(0) \langle f, \eta(x) x^\alpha \rangle = \sum_{|\alpha| \leq k} c_\alpha (\partial^\alpha \delta_0)(\phi),$$

where $c_\alpha = (-1)^{|\alpha|} \langle f, \eta(x) x^\alpha \rangle / \alpha!$.

□
